

In this project we shall construct some multiply transitive groups starting from a representation of  $S_6$ . Let

$$Z = \{a, b, c, d, e, f\}$$

and let  $X$  denote the set of 10 partitions of  $Z$  into two sets of three. Label the members of  $X$  as follows:

$$\begin{array}{ll} 0 & abc | def \\ 1 & abd | cef \\ 2 & abe | cdf \\ 3 & abf | cde \\ 4 & acd | bef \\ 5 & ace | bdf \\ 6 & acf | bde \\ 7 & ade | bcf \\ 8 & adf | bce \\ 9 & aef | bcd \end{array}$$

1. Let  $g \mapsto \hat{g}$  denote the representation of  $S_6 = \text{Sym}(Z)$  as permutations of  $X$ . Check that it is a faithful representation and that  $\widehat{S_6}$  acts transitively on  $X$ . Find the orders:

- i. the pointwise stabilizer of  $abc$  in  $S_6$ ;
- ii. the setwise stabilizer of  $abc$  in  $S_6$ ;
- iii. the stabilizer of 0 in  $\widehat{S_6}$ .

In general, we'll use the following properties. Let  $f : X \rightarrow Y$  be a bijection and  $A, B \subseteq X$ . Then

$$|A \cap B| = |f(A) \cap f(B)|.$$

*Proof.* 1. We first prove that

$$f(A \cap B) = f(A) \cap f(B).$$

If  $x \in A \cap B$ , then  $f(x) \in f(A)$  and  $f(x) \in f(B)$ , hence  $f(x) \in f(A) \cap f(B)$ . Thus,

$$f(A \cap B) \subseteq f(A) \cap f(B).$$

Conversely, suppose  $y \in f(A) \cap f(B)$ . Then there exist  $a \in A$  and  $b \in B$  such that  $f(a) = y = f(b)$ . Since  $f$  is injective,  $a = b$ , so  $a \in A \cap B$  and  $y = f(a) \in f(A \cap B)$ . Hence,

$$f(A) \cap f(B) \subseteq f(A \cap B).$$

Therefore,

$$f(A \cap B) = f(A) \cap f(B).$$

2. From the previous step,

$$f(A) \cap f(B) = f(A \cap B).$$

Now the restriction

$$f|_{A \cap B} : A \cap B \longrightarrow f(A \cap B)$$

is a bijection. Hence  $A \cap B$  and  $f(A \cap B)$  have the same cardinality:

$$|A \cap B| = |f(A \cap B)| = |f(A) \cap f(B)|.$$

In particular, if  $|A \cap B| = n \in \mathbb{N}$ , then  $|f(A) \cap f(B)| = n$ .

□

*Proof.* For each partition  $i$ , write  $\{P_i \mid P'_i\}$ , and let  $g \in S_6$  such that  $\hat{g}$  fixes all partitions as elements of  $X$ .

Suppose  $gP_0 = P'_0$ .

Since  $|P_0 \cap P_1| = 2$ , we must have  $|g(P_0) \cap g(P_1)| = 2$ . Hence

$$gP_1 = P'_1.$$

But since  $|P_0 \cap P_3| = 2$ , we must have  $|gP_0 \cap gP_3| = 2$ , while  $|P'_0 \cap P_3| = |P'_0 \cap P'_3| = 1$ , a contradiction. Therefore

$$gP_0 = P_0.$$

For  $k = 1, \dots, 6$ ,  $|P_0 \cap P_k| = 2$ . If some  $P_k$  were mapped to its complement  $P'_k$ , then

$$|gP_0 \cap gP_k| = |P_0 \cap P'_k| = 1,$$

contradiction. Hence

$$gP_k = P_k \quad (k = 1, \dots, 6).$$

From  $gP_0 = P_0$  and  $gP_6 = P_6$ , we deduce

$$g(a) \in gP_0 \cap gP_6 = P_0 \cap P_6 = \{a\}.$$

Since  $g(a) = a$  and  $gP_0 = P_0$ , we get  $g(b) \in \{b, c\}$ . As  $b \in P_1$  and  $gP_1 = P_1$ , then

$$g(b) \in P_1 \cap \{b, c\} = \{b\}.$$

1. From  $gP_0 = P_0 = \{a, b, c\}$  and  $g(a) = a$ ,  $g(b) = b$ , we deduce  $g(c) = c$ .
2. From  $gP_1 = P_1 = \{a, b, d\}$  and  $g(a) = a$ ,  $g(b) = b$ , we deduce  $g(d) = d$ .
3. From  $gP_2 = P_2 = \{a, b, e\}$  and  $g(a) = a$ ,  $g(b) = b$ , we deduce  $g(e) = e$ .

4. Finally, we deduce  $g(f) = f$ .

Thus  $g = \text{id}$ , so the representation is faithful.

Since  $S_6$  acts 3-transitively on  $Z$ , let  $\{A \mid A^c\}, \{B \mid B^c\} \in X$ . There exists  $g \in S_6$  such that  $g(A) = B$ , hence  $g(A^c) = B^c$ , so  $g \cdot \{A \mid A^c\} = \{B \mid B^c\}$ .

1. Fixing  $\{a, b, c\}$  pointwise means permuting freely  $\{d, e, f\}$ , giving a subgroup isomorphic to  $S_3$ , of order  $|S_3| = 6$ .
2. Fixing  $\{a, b, c\}$  setwise allows independent permutations of  $\{a, b, c\}$  and  $\{d, e, f\}$ , yielding  $S_3 \times S_3$ , of order  $|S_3 \times S_3| = 36$ .
3. The stabilizer of the partition  $\{P_0 \mid P'_0\}$  includes the above automorphisms plus the symmetry exchanging  $P_0$  and  $P'_0$ . This doubles the size, giving order 72.

□

2. By considering  $(abc)^\wedge$  and  $(def)^\wedge$ , show that  $\widehat{S}_6$  is 2-transitive on  $X$ .

*Proof.* Let  $g = (abc)^\wedge$  and  $h = (def)^\wedge$ . Then

$i$	0	1	2	3	4	5	6	7	8	9
$g(i)$	0	9	8	7	1	2	3	6	5	4
$i$	0	1	2	3	4	5	6	7	8	9
$h(i)$	0	2	3	1	5	6	4	9	7	8

From these tables:

$$\varphi_1 = g = (194)(285)(376), \quad \varphi_2 = h = (123)(456)(798),$$

with 0 fixed in both cases. Therefore

$$\langle g, h \rangle(1) = \{1, 2, 3, 4, 5, 6, 7, 8, 9\} = X \setminus \{0\}.$$

Hence the stabilizer of 0 is transitive on  $X \setminus \{0\}$ . Since  $\widehat{S}_6$  is transitive on  $X$ , the action is 2-transitive. □

3. How many elements in  $\widehat{S}_6$  fix both 0 and 1? Find them. Deduce that  $\widehat{S}_6$  is not 3-transitive on  $X$ .

*Proof.* An element  $g \in S_6$  fixes 0 iff it sends the partition  $\{abc \mid def\}$  to itself (possibly interchanging the blocks). Consider two cases.

**(I)  $g$  preserves each triple  $\{a, b, c\}$  and  $\{d, e, f\}$ .** To also fix  $1 = \{abd \mid cef\}$ , necessarily

$$\{a, b\} \mapsto \{a, b\}, \quad d \mapsto d, \quad c \mapsto c, \quad \{e, f\} \mapsto \{e, f\}.$$

This gives exactly 4 elements:

$$\text{id}, (ab), (ef), (ab)(ef).$$

**(II)  $g$  interchanges  $\{a, b, c\} \leftrightarrow \{d, e, f\}$ .** To fix 1 as a partition, we must have

$$d \mapsto c, \quad \{a, b\} \mapsto \{e, f\}, \quad \{e, f\} \mapsto \{a, b\}.$$

This produces 4 more elements:

$$(ae)(bf)(cd), (af)(be)(cd), (aebf)(cd), (afbe)(cd).$$

Hence the stabilizer of  $\{0, 1\}$  in  $\widehat{S}_6$  has 8 elements:

$$\{\text{id}, (ab), (ef), (ab)(ef), (ae)(bf)(cd), (af)(be)(cd), (aebf)(cd), (afbe)(cd)\}.$$

Among them, those also fixing  $2 = \{abe \mid cdf\}$  are

$$\text{id}, (ab).$$

Thus the 3-point stabilizer has size  $2 > 1$ . If the action were 3-transitive, then by orbit-stabilizer:

$$|\widehat{S}_6| = |X| \cdot (|X| - 1) \cdot (|X| - 2) \cdot |(\widehat{S}_6)_{0,1,2}|.$$

Since  $|X| = 10$  and  $|\text{Stab}(0, 1, 2)| = 2$ , we would obtain

$$|\widehat{S}_6| = 10 \cdot 9 \cdot 8 \cdot 2 = 1440,$$

which contradicts  $|\widehat{S}_6| = |S_6| = 720$ . Therefore the action is not 3-transitive.  $\square$

4. Let  $H = \widehat{A}_6$  be the group of permutations of  $X$  induced by the even permutations of  $Z$ . Verify that  $H$  is 2-transitive on  $X$  and that  $H_{01}$  is a cyclic group of order 4 generated by

$$\theta = (afbe)(cd)^\wedge = (2934)(5876).$$

*Proof.* The elements  $g = (abc)^\wedge$  and  $h = (def)^\wedge$  are even, belong to  $H$ , and fix 0. Since we already verified that  $\langle g, h \rangle$  is transitive on  $X \setminus \{0\}$ ,  $H_0$  is transitive. As  $H$  acts transitively on  $X$ , the action of  $H$  is 2-transitive. For  $H_{01}$ , among the 8 elements of  $\widehat{S}_6$  fixing 0 and 1, the even ones are exactly

$$\theta = (afbe)(cd), \quad \theta^2 = (ab)(ef), \quad \theta^3 = (aebf)(cd), \quad \theta^4 = \text{id}.$$

Hence  $H_{01} = \langle \theta \rangle \cong C_4$ .  $\square$

5. Show that  $H_0$  is generated by  $\theta$ ,  $\varphi_1 = (abc)^\wedge$  and  $\varphi_2 = (def)^\wedge$ . Use the fact that  $H$  is primitive on  $X$  to deduce that  $H$  is generated by  $\theta, \varphi_1, \varphi_2$ , and any element  $\psi$  of  $H - H_0$ .

*Proof.* By the previous theorem

$$H_0 = H_{01} \cup \bigcup_{y \neq 1} g_y H_{01},$$

where  $g_y$  are representatives of the class  $G(1 \mapsto y)$

In exercise 4 we proved that

$$H_{01} = \langle \theta \rangle.$$

Choosing  $g_y \in \langle \varphi_1, \varphi_2 \rangle$

$$H_0 = \langle \theta \rangle \cup \bigcup_{y \neq 1} (g_y H_{01}) \subseteq \langle \theta, \varphi_1, \varphi_2 \rangle.$$

Since  $H$  is primitive by theorem 1.6.5 we must have that the pointwise stabilizer are maximal subgroups so

$$H_0 = \langle \theta, \varphi_1, \varphi_2 \rangle.$$

By Lemma 1.3.9, since  $H$  is 2-transitive and  $\psi \notin H_0$ , we have

$$H = H_0 \cup H_0 \psi H_0 \subseteq \langle \theta, \varphi_1, \varphi_2, \psi \rangle.$$

$\square$

6. Take  $\psi$  to be  $(ab)(cd)^\wedge = (01)(49)(56)(78)$ , so that

$$H = \langle \theta, \varphi_1, \varphi_2, \psi \rangle.$$

Define a new permutation, not in  $H$ , by

$$\lambda = (2735)(4698).$$

Show that the conjugate of each generator of  $H$  by  $\lambda$  is an element of  $H$  and that  $\lambda^2 \in H$ . Deduce that there are just two cosets of  $H$  in  $\langle H, \lambda \rangle$ .

*Proof.* Suppose  $\lambda = g^\wedge$  for some  $g \in S_6$  since  $\lambda(2) = 7$  we must have

$$g(\{a, b, e\}) = \begin{cases} \{a, d, e\} \\ \{b, c, f\} \end{cases}$$

analogously

$$g(\{a, d, e\}) = \begin{cases} \{a, b, f\} \\ \{c, d, e\} \end{cases}$$

$$g(\{a, b, f\}) = \begin{cases} \{a, c, e\} \\ \{b, d, f\} \end{cases}$$

$$g(\{a, c, e\}) = \begin{cases} \{a, b, e\} \\ \{c, d, f\} \end{cases}$$

But  $2 = |\{a, b, e\} \cap \{a, d, e\} \cap \{a, c, e\}|$  and no choice of the images of these three sets gives us a combination of sets which intersect in 2 points. Therefore  $\lambda \notin \widehat{S}_6$ , in particular,  $\lambda \notin \widehat{A}_6 = H$ . Checking the conjugations by  $\lambda$ :

$$\lambda \varphi_1 \lambda^{-1} = (186)(742)(539) = (abc)(def),$$

$$\lambda \varphi_2 \lambda^{-1} = (175)(629)(384) = (abc)(dfe),$$

$$\lambda \psi \lambda^{-1} = (01)(68)(29)(34) = (ae)(bf),$$

$$\lambda \theta \lambda^{-1} = (7856)(2439) = (aebf)(cd) = \theta^{-1}.$$

Also,

$$\lambda^2 = (23)(75)(49)(68) = (ab)(ef).$$

Thus the cosets look like  $H, H\lambda^r$ . Since  $\lambda^2 \in H$ , the only cosets are  $H$  and  $H\lambda$ .  $\square$

7. Define  $M_{10}$  (the Mathieu group on 10 symbols) to be

$$M_{10} = \langle H, \lambda \rangle.$$

Verify that  $M_{10}$  is sharply 3-transitive on  $X$ .

*Proof.* Since  $\lambda$  fixes 0 and 1, we have  $\lambda \in M_{10,01}$ . Hence

$$\langle \theta, \lambda \rangle \leq M_{10,01}.$$

For the reverse inclusion, let  $g \in M_{10,01}$ . Because  $[M_{10} : H] = 2$ , we can write

$$g \in H \quad \text{or} \quad g \in H\lambda.$$

If  $g \in H$  and  $g$  fixes 0, 1, then  $g \in H_{01} = \langle \theta \rangle \leq \langle \theta, \lambda \rangle$ . If  $g \in H\lambda$ , write  $g = h\lambda$  with  $h \in H$ . Since  $\lambda(0) = 0$  and  $\lambda(1) = 1$ , the equalities

$$0 = g(0) = h(\lambda(0)) = h(0), \quad 1 = g(1) = h(\lambda(1)) = h(1)$$

force  $h \in H_{01} = \langle \theta \rangle$ . Thus  $g = h\lambda \in \langle \theta \rangle\lambda \subseteq \langle \theta, \lambda \rangle$ .

Therefore every  $g \in M_{10,01}$  lies in  $\langle \theta, \lambda \rangle$ , so  $M_{10,01} = \langle \theta, \lambda \rangle$  and  $\lambda\theta\lambda^{-1} = \theta^{-1}$ , so  $\langle \theta \rangle \triangleleft M_{10,01}$ . Since  $\lambda^2 \in \langle \theta \rangle$ , the elements of  $M_{10,01}$  are of the form

$$\theta^i \quad \text{or} \quad \lambda\theta^i, \quad i = 0, 1, 2, 3.$$

Finally, the orbit of 2 under  $\langle \theta, \lambda \rangle$  is all of  $X \setminus \{0, 1\}$ . Hence  $M_{10,01}$  is transitive on  $X \setminus \{0, 1\}$  and

$$|M_{10,01}(2)| = \frac{|M_{10,01}|}{|(M_{10,01})_2|}.$$

As the orbit of 2 is all  $X \setminus \{0, 1\}$ , we obtain

$$8 = \frac{8}{|(M_{10,01})_2|} \quad \Rightarrow \quad |(M_{10,01})_2| = 1.$$

Hence

$$|M_{10,(0,1,2)}| = 1,$$

i.e. the stabilizer of  $(0, 1, 2)$  in  $M_{10}$  is trivial.  $\square$

8. In 1.5.2, let  $G = M_{10}$ ,  $* = T$ , and

$$h = (0T)(47)(59)(68), \quad g = \psi.$$

Show that

$$G_0 = \langle \theta, \varphi_1, \varphi_2, \lambda \rangle$$

and verify that the conditions of the theorem hold, so that

$$M_{11} = \langle M_{10}, h \rangle$$

is sharply 4-transitive on  $X' = \{0, 1, \dots, 9, T\}$ .

*Proof.* We know that

$$\langle H_0, \lambda \rangle \leq G_0.$$

For the reverse inclusion, let  $g \in G_0$ . Since  $M_{10} = H \cup H\lambda$ , either  $g \in H$  or  $g \in H\lambda$ . If  $g \in H$  and  $g(0) = 0$ , then  $g \in H_0 \leq \langle H_0, \lambda \rangle$ . If  $g \in H\lambda$ , write  $g = h\lambda$  with  $h \in H$ . As  $\lambda(0) = 0$  and  $g(0) = 0$ ,

$$0 = g(0) = h(\lambda(0)) = h(0),$$

so  $h \in H_0$  and thus  $g = h\lambda \in \langle H_0, \lambda \rangle$ .

Therefore  $G_0 = \langle H_0, \lambda \rangle = \langle \theta, \varphi_1, \varphi_2, \lambda \rangle$ .

Now, for the one point extension

(i)  $h$  interchanges  $T$  and  $x = 0$  and fixes  $y = 1$ :

$$h(T) = 0, \quad h(0) = T, \quad h(1) = 1.$$

(ii)  $g$  interchanges  $x$  and  $y$ :

$$g(0) = 1, \quad g(1) = 0.$$

(iii) We have  $h^2 = \text{id} \in G$  and

$$gh = (0T1)(485)(679),$$

so that

$$(gh)^3 = \text{id} \in G.$$

(iv) Since  $h$  has order 2,  $h^{-1} = h$ :

$$* h\theta h = (2537)(9648) = \lambda^{-1}.$$

$$* h\varphi_2 h = (123)(456)(798) = \varphi_2.$$

$$* h\lambda h = (2439)(7856) = \theta^{-1}.$$

$$* h\varphi_1 h = (157)(269)(348) = \lambda\varphi_1\lambda^{-1}\varphi_1.$$

Therefore  $hG_0h = G_0$ , and with this and Theorem 1.5.2 we conclude that  $M_{11}$  is a one-point extension of  $M_{10}$ . Since  $M_{10}$  was sharply 3-transitive,  $M_{11}$  is sharply 4-transitive.  $\square$

9. Now, let  $G' = M_{11}$ ,  $*' = E$ , and  $g' = h$ . Find a permutation  $h'$  of  $\{0, 1, \dots, 9, T, E\}$  such that

$$M_{12} = \langle M_{11}, h' \rangle$$

is sharply 5-transitive.

*Proof.* Let

$$G' = M_{11}, \quad *' = E, \quad g' = h = (0T)(47)(59)(68).$$

Define

$$h' = (TE)(49)(58)(67).$$

We also set

$$x := T, \quad y := 0.$$

(i) By definition  $h'(E) = T$ ,  $h'(T) = E$ , and  $h'(0) = 0$ .

(ii) Also by construction we have  $g'(T) = 0$  and  $g'(0) = T$ .

(iii) The permutation  $h'$  is a transposition, hence  $h'^2 = \text{id}$  and

$$g'h' = (0T)(47)(59)(68) \cdot (TE)(49)(58)(67) = (0TE)(456)(789)$$

so  $(g'h')^3 = \text{id}$ .

(iv) Since  $M_{11}$  is a 1 point extension of  $M_{10}$   $M_{11,T} = M_{10}$ . Denotamos  $\sigma = (49)(58)(67)$ . Como el dominio de  $h'$  solo intersecta al de los generadores en  $\{4, 5, 6, 7, 8, 9\}$ , basta verificar las igualdades con  $\sigma$ .

1. Para  $\psi = (01)(49)(56)(78)$ :

$$\sigma\psi\sigma^{-1} = (01)(49)(56)(78) = \psi.$$

2. Para  $\varphi_2 = (123)(456)(798)$ :

$$\sigma\varphi_2\sigma^{-1} = (123)(\sigma(4)\sigma(5)\sigma(6))(\sigma(7)\sigma(9)\sigma(8)) = (123)(456)(798) = \varphi_2.$$

3. Para  $\theta = (2934)(5876)$ :

$$\sigma\theta\sigma^{-1} = (2\sigma(9)3\sigma(4))(\sigma(5)\sigma(8)\sigma(7)\sigma(6)) = (2439)(5678) = \theta^{-1}.$$

4. Para  $\varphi_1 = (194)(285)(376)$ :

$$\sigma\varphi_1\sigma^{-1} = (1\sigma(9)\sigma(4))(2\sigma(8)\sigma(5))(3\sigma(7)\sigma(6)) = (149)(258)(367) = \varphi_1^{-1}.$$

5. Para  $\lambda = (2735)(4698)$ :

$$\sigma\lambda\sigma^{-1} = (2\sigma(7)3\sigma(5))(\sigma(4)\sigma(6)\sigma(9)\sigma(8)) = (2638)(4597) = \lambda^{-1}\theta.$$

Cada conjugado resulta una palabra en los generadores de  $M_{10}$ . Por lo tanto,  $h'$  normaliza  $M_{10}$ .  $\square$

10. Suppose  $Z$  is replaced by a set with 8, 10, ... members; at what stage does the preceding construction break down?

*Proof.* Let  $Z = \{a, b, c, d, e, f, g, h\}$  and let  $X$  be the set of partitions of  $Z$  into two 4-subsets.

Fix

$$0 := \{P_0 \mid P'_0\} \quad \text{with} \quad P_0 = \{a, b, c, d\}, \quad P'_0 = \{e, f, g, h\}.$$

Define

$$1 := \{P_1 \mid P'_1\} \quad \text{with} \quad P_1 = \{a, e, f, g\}, \quad P'_1 = \{b, c, d, h\},$$

$$2 := \{P_2 \mid P'_2\} \quad \text{with} \quad P_2 = \{a, b, e, f\}, \quad P'_2 = \{c, d, g, h\}.$$

Let  $g \in \text{Stab}(0)$ . Then  $g(P_0) = P_0$  or  $g(P_0) = P'_0$ . In either case,

$$|g(P_0) \cap P_2| = |g(P_0) \cap P'_2| = 2.$$

Suppose  $g(1) = 2$ . Then we have two options:  $g(P_1) = P_2$  or  $g(P_1) = P'_2$ . In either case

$$1 = |P_0 \cap P_1| = |g(P_0) \cap g(P_1)| = 2,$$

a contradiction. Therefore the action is not 2-transitive once  $|Z| \geq 8$ .  $\square$

11. Is  $M_{10}$  simple? Assume that  $M_{11}$  is simple, and show that  $M_{12}$  is simple.

*Proof.* From Exercises 6–8 we have  $H \triangleleft M_{10}$  is a proper nontrivial normal subgroup, so  $M_{10}$  is not simple.

Let  $G := M_{12}$  acting sharply 5-transitively on  $\Omega = \{0, 1, \dots, 9, T, E\}$  and for any  $\alpha \in \Omega$  the point stabilizer  $G_\alpha \cong M_{11}$  has order  $|G_\alpha| = 11 \cdot 10 \cdot 9 \cdot 8 = 7920$  and is simple.

Let  $1 \neq N \triangleleft G$ . Since  $G$  is 5-transitive it is primitive (1.6.3) and  $N$  is transitive on  $\Omega$  (1.6.6).

By 1.6.7, either

- (i)  $G$  is simple, or
- (ii)  $G$  has a normal subgroup  $N \triangleleft G$  acting regularly on  $\Omega$ .

Suppose (ii) holds. Then  $|N| = |\Omega| = 12$ . Since  $11 \mid |G_\alpha|$ , choose  $t \in G_\alpha$  of order 11. Conjugation by  $t$  induces an automorphism of  $N$  of order 11. But no group  $N$  of order 12 has an automorphism of order 11 [https://groupprops.subwiki.org/wiki/Groups\\_of\\_order\\_12#Automorphism\\_groups](https://groupprops.subwiki.org/wiki/Groups_of_order_12#Automorphism_groups). Hence (ii) is impossible, and we must have (i):  $G = M_{12}$  is simple.  $\square$